https://www.linkedin.com/feed/update/urn:li:activity:6477418861909151744

$\lim_{n\to\infty}\left(\sqrt[n]{1}+\sqrt[n]{2}+...+\sqrt[n]{2007}-2006\right)^n$

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We will find $\lim_{n\to\infty}\left(\sum_{k=1}^{m+1}\sqrt[n]{k}-m\right)^n$, for any given $m\in\mathbb{N}$.

Let $a_n:=\sum_{k=1}^{m}\left(\sqrt[n]{k+1}-1\right)$. Since $\lim_{n\to\infty}a_n=\sum_{k=1}^{m}\lim_{n\to\infty}\left(\sqrt[n]{k+1}-1\right)=0$ (because $\lim_{n\to\infty}\sqrt[n]{k+1}=1, k\in 1,2,...,m$) and $\sum_{k=1}^{m+1}\sqrt[n]{k}-m=1+a_n$ then $\lim_{n\to\infty}\left(\sum_{k=1}^{m+1}\sqrt[n]{k}-m\right)^n=\lim_{n\to\infty}(1+a_n)^n=\lim_{n\to\infty}\left((1+a_n)^{\frac{1}{a_n}}\right)^{n\cdot a_n}$.

Noting that $\lim_{n\to\infty}(1+a_n)^{\frac{1}{a_n}}=e$ and $\lim_{n\to\infty}n\cdot a_n=\sum_{k=1}^{m}\lim_{n\to\infty}n\left(\sqrt[n]{k+1}-1\right)=\sum_{k=1}^{m}\ln(k+1)=\ln((m+1)!)$ (because * $\lim_{n\to\infty}n\left(\sqrt[n]{a}-1\right)=\ln a$)

we obtain $\lim_{n\to\infty}\left(\sum_{k=1}^{m+1}\sqrt[n]{k}-m\right)^n=e^{\ln((m+1)!)}=(m+1)!$.

In particular for $m=2006$ we have

$\lim_{n\to\infty}\left(\sqrt[n]{1}+\sqrt[n]{2}+...+\sqrt[n]{2007}-2006\right)^n=2007!$.

* Let $\alpha_n:=\dfrac{\ln a}{n}$.Since $\lim_{n\to\infty}\alpha_n=0$ then $\lim_{n\to\infty}n\left(\sqrt[n]{a}-1\right)=\lim_{n\to\infty}n\left(e^{\frac{\ln a}{n}}-1\right)=\ln a\cdot\lim_{n\to\infty}\dfrac{e^{\alpha_n}-1}{\alpha_n}=\ln a$.